A COUNTABLE SERIES OF BISIMPLE $\mathcal{H}$-TRIVIAL FINITELY PRESENTED CONGRUENCE-FREE MONOIDS

VICTOR MALTCEV

It is well-known a countable series of infinite finitely presented simple groups by Graham Higman [6]. In view of that Boone-Higman Conjecture (see [7] to learn about the results on finitely generated simple groups) is still an open problem, every example of finitely generated (and especially finitely presented) group and semigroup are very precious. Surprisingly, the semigroup analogue of simple groups – congruence-free semigroups – were studied not that extensively as that was for simple groups. The only results in this area are due to Jean-Camille Birget, we refer the reader to [1]–[5] to learn about semigroup analogues of Higman's groups.

The author of the current note asked himself – would it be possible to find the 'really' semigroup counterpart of Higman's series, i.e. so that each of the semigroups is finitely presented, congruence-free, bisimple, but yet $\mathcal{H}$-trivial. Indeed such a series exists and the main goal of the note is to prove

Main Result. *For any $n \geq 1$ the monoid M_n presented by the (finite) confluent noetherian system*

$$\begin{array}{rcl} a^{n+1}ba^nb & \to & 1 \\ a^{n+1}ba^{n-1}b & \to & 1 \\ & \cdots & \\ a^{n+1}bab & \to & 1 \\ a^{n+1}b^2 & \to & b \end{array}$$

is bisimple, $\mathcal{H}$-trivial and congruence-free.

Proof. Since M_n is given by a finite complete system, it is convenient to work with elements of M_n in their normal forms. So, a typical element of M_n looks like

$$b^{\varepsilon}(a^{d_1}b^{k_1})\cdots(a^{d_s}b^{k_s})(a^{n_1}b)\cdots(a^{n_p}b)a^l, \tag{1}$$

where $\varepsilon \geq 0$, $s \geq 0$, $p \geq 0$, $l \geq 0$; $1 \leq d_1, \ldots, d_s \leq n$; $k_1, \ldots, k_s \geq 1$; $n_1, \ldots, n_p \geq n+1$.

Later we will need the following norm for the elements of M_n: if $w \in M_n$ admits the normal form (1), then

$$\|w\| = \varepsilon + k_1 + \cdots + k_s + n_1 \cdots + n_p + l.$$

Bisimplicity and $\mathcal{H}$-triviality

It is straightforward from the presentation that $a^l\mathcal{R}1$ for all $l \geq 0$, and that $a^mb\mathcal{R}1$ for all $m \geq n+1$. Furthermore, $b^{\varepsilon}\mathcal{L}1$ for all $\varepsilon \geq 0$, and $a^db^k\mathcal{L}1$ for all $1 \leq d \leq n$ and $k \geq 1$. Using these, by (1) we immediately see that every element of

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M_n is $\mathcal{D}$-equivalent to 1 and so M_n is bisimple. One can quite easily assure oneself that M_n is, in addition, $\mathcal{H}$-trivial.

Congruence-freeness

That M_n is congruence-free is equivalent to the following: for every congruence ρ on M_n, and $u \neq v$ in M_n such that $u\rho v$, it follows that $\rho = M_n \times M_n$.

We induct on $\|u\| + \|v\|$ to show that if $u \neq v$ in M_n and ρ is a congruence on M_n with $u\rho v$, then $\rho = M_n \times M_n$. Prior to starting our induction we prove two facts we will use frequently within the induction arguments:

Lemma 1. *Any group homomorphic image of M_n is trivial.*

Proof. Let G be the group presented by the initial presentation for M_n. Then $a^{n+1}b = 1$ and so $a^n b = 1$ in G. This yields $a = 1$ and $b = 1$ in G. □

Lemma 2. *Let ρ be a congruence on M_n such that $a^d b\mathcal{R}1$ in M_n/ρ for some $1 \leq d \leq n$. Then $\rho = M_n \times M_n$.*

Proof. Since $a^{n+1}ba^d b = 1$, we have that $a^d b \in H_1$ in M_n/ρ. Again by $a^{n+1}ba^d b = 1$, then $a^{n+1}b \in H_1$ and so $a^{n+1-d} \in H_1$ (in M_n/ρ). This yields $a \in H_1$ and thus $b^2 \in H_1$ (in M_n/ρ). Therefore $a, b \in H_1$ (in M_n/ρ) and thus M_n/ρ is a group. By Lemma 1, $\rho = M_n \times M_n$. □

Base of induction: $\|u\| + \|v\| = 1$

Without loss we may assume that $\|u\| = 1$ and $v = 1$. The possibilities for u are:

- $u = b$. Then $b\rho 1$ and so $a^{n+1}\rho 1$ and $a^{n+2}\rho 1$. Thus $a\rho 1$ and so $\rho = M_n \times M_n$.
- $u = a$. Then $a\rho 1$ and so $b^2\rho 1$ and $b^2\rho b$. This implies $b\rho 1$ and thus $\rho = M_n \times M_n$.
- $u = a^d b$ with $1 \leq d \leq n$. Then by Lemma 2, $\rho = M_n \times M_n$.

Induction step: $\big(< \|u\| + \|v\|\big) \longmapsto \big(\|u\| + \|v\|\big)$

We may assume that u and v are in their normal forms. First let us sort out the case when one of u and v is the identity. Let, say, $v = 1$. If u starts with b, then $b \in H_1$ in M_n/ρ. Then by $a^{n+1}b^2 = b$, $a \in H_1$ in M_n/ρ. By Lemma 1, then we have $\rho = M_n \times M_n$. If u starts with a, then three cases can happen:

- $u \equiv a^d bU$ for $1 \leq d \leq n$. Then $a^d b\mathcal{R}1$ in M_n/ρ and so by Lemma 2 $\rho = M_n \times M_n$.
- $u \equiv Ua^m b$ for $m \geq n+1$. Then $ab\rho Ua^m bab = Ua^{m-n-1}$ and so $aba^{n+1}b\rho 1$. Then $ab\mathcal{R}1$ in M_n/ρ and we are done.
- $u \equiv Ua^m ba^l$ for $m \geq n+1$ and $l \geq 1$. If $l \geq n$, then

$$a^{n+1}bUa^m ba^{l+1}b\rho a^{n+1}bab = 1$$

  and running the arguments from the previous case, we derive that $\rho = M_n \times M_n$. If $l \leq n-1$, then $Ua^m b\rho a^{n+1-l}bab$. This yields $a^{n+1-l}baba^l\rho Ua^m ba^l\rho 1$, and so $a^{n+1-l}b\mathcal{R}1$ in M_n/ρ, thus $\rho = M_n \times M_n$.

From now on, in the remainder of the induction step we may assume that $u \neq 1$ and $v \neq 1$. Now we sort out the case when at least one of u and v starts with b. Essentially there are only three following cases:

- $u \equiv bU$ and $v \equiv bV$. Then $U \not\equiv V$ and so $U \neq V$ (since U and V in their normal forms). But $b\mathcal{L}1$ and so $U\rho V$. By induction, $\rho = M_n \times M_n$.
- $u \equiv b^\varepsilon U$ and $v = a^d b^k V$ for $1 \leq d \leq n$, $\varepsilon \geq 1$, $k \geq 1$. Then $b^\varepsilon U = a^{n+1}b \cdot b^\varepsilon U \rho b^{k-1}V$. Note that $b^\varepsilon U$ and $b^{k-1}V$ are in their normal forms. Thus if $b^\varepsilon U \not\equiv b^{k-1}V$, by induction it follows that $\rho = M_n \times M_n$. So, assume that $b^\varepsilon U \equiv b^{k-1}V$. Then $k \geq 2$ and so $a^{n+1-d}b^{k-1}V\rho a^{n+1}b^2 \cdot b^{k-2}V = b^{k-1}V$. Now by induction, $\rho = M_n \times M_n$.
- $u \equiv b^\varepsilon U$ and $v \equiv a^m bV$ for $\varepsilon \geq 1$ and $m \geq n+1$. Then $b\mathcal{R}1$ in M_n/ρ, and so $b \in H_1$ in M_n/ρ. By $a^{n+1}b^2 = b$, this gives $a \in H_1$ in M_n/ρ and so by Lemma 1, $\rho = M_n \times M_n$.

We are left with the situation when both u and v start with a. First we deal with the case when one of u and v starts with a prefix of the form $a^d b$ for some $1 \leq d \leq n$. Up to changing the roles of u and v, this splits into the following two cases:

- $u \equiv a^{d+r}b^k U$ and $v \equiv a^d b^t V$ for $1 \leq d, d+r \leq n$, and $d, r, k, t \geq 1$, and such that U is either empty or starts with a. Premultiplying $u\rho v$ with $a^{n+1}ba^{n-d-r}$, we obtain $b^{k-1}U\rho b^{t-1}V$. Both $b^{k-1}U$ and $b^{t-1}V$ are in their normal forms. If $b^{k-1}U \not\equiv b^{t-1}V$, then by induction we are done. So, assume that $b^{k-1}U \equiv b^{t-1}V$. Then $a^{d+r}b^k U\rho a^d b^k U$. If $k > 1$, then by premultiplying the last ρ-equivalence with $a^{n+1-d-r}$, we obtain $b^{k-1}U\rho a^{n+1-r}b^k U$ and can apply induction. So assume that $k = 1$. Then $a^{n+1}bU\rho a^{n+1-r}bU$ and so in general $a^{n+1+pr}bU\rho a^{n+1-r}bU$ for all $p \geq 0$. For a sufficiently large p, $a^{n+1+pr}bU\mathcal{R}1$ and so $a^{n+1-r}b\mathcal{R}1$ in M_n/ρ. By Lemma 2, then $\rho = M_n \times M_n$.
- $u \equiv a^d b^k U$ and $v \equiv a^m bV$ for $1 \leq d \leq n$, $k \geq 1$, $m \geq n+1$; and such that U is either empty or starts with a. If $k \geq 2$, then $b^{k-1}U\rho a^{n+1-d+m}bV$ and so $b\mathcal{R}1$ in M_n/ρ. Then, as above, $a\rho b\rho 1$ and so $\rho = M_n \times M_n$. If $k = 1$, then $a^d bU\rho a^m bV$ and so $U\rho a^{n+1}ba^m bV$. Hence $a^d ba^{n+1}ba^m bV\rho a^m bV$ and so $a^d ba^{n+1}b\rho 1$. Therefore $a^d b\mathcal{R}1$ in M_n/ρ and by Lemma 2 we are done.

In the remainder of the induction step, both u and v will start with prefixes of the form $a^m b$ for $m \geq n+1$. If both u and v end with a, we can cancel this distinguished a (since $a\mathcal{R}1$) and then apply induction. So without loss we will assume that $u \equiv Ua^m b$ (with $m \geq n+1$). We have two cases to consider depending on whether v ends with a or not:

- $v \equiv Va^p b$ with $p \geq n+1$. Then postmultiplying $Ua^m b\rho Va^p b$ with ab, we obtain $Ua^{m-n-1}\rho Va^{p-n-1}$ and may apply induction.
- $v \equiv Va^l$ with $l \geq 1$. Then $Ua^m b\rho Va^l$ and so, by postmultiplying with $a^n b$, $Ua^{m-n-1}\rho Va^{l+n}b$. If $Ua^{m-n-1} \not\equiv Va^{l+n}b$, then we may apply induction. So assume that $Ua^{m-n-1} \equiv Va^{l+n}b$. Then $m = n+1$ and $u\rho v$ reads as

$$Va^{l+n}ba^{n+1}b\rho Va^l. \tag{2}$$

  If $l \geq n+1$, then $Va^l b\rho Va^{l+n}b^2 = Va^{l-1}b$, and since $Va^l b$ is in its normal form (and regardless whether $Va^{l-1}b$ must be reduced to get its normal form), we may apply induction. So we will assume that $l \leq n$. If $2 \leq l \leq n$, then postmultiply (2) with a^{n+1-l} to obtain $Va^{n+1}b\rho Va^{l+n}b$ and then use

induction. Hence let $l=1$, then posmultiply (2) with b: $Vab\rho Va^{n+1}b^2 = Vb$. If $V \equiv 1$, then $ab\rho b$ and so $b^2\rho 1$ and $a\rho ab^2\rho b^2\rho 1$. This yields that M_n/ρ is a group and so we are done. Thus we may assume that V is non-empty: $V \equiv V'a^tb$ (with $t \geq n+1$). Then

$$V'a^{t-n-1} = V'a^tbab\rho V'a^tb^2 = V'a^{t-n-1}b.$$

If $t-n-1 \geq n+1$, then we may apply induction. So let $t-n-1 \leq n$. If $V' \equiv 1$, then $a^n\rho a^nb$ and so $1\rho a^{n+1}ba^n$. This implies $a \in H_1$ in M_n/ρ, which in turn gives $b \in H_1$ in M_n/ρ. These imply $\rho = M_n \times M_n$. Thus we may assume that V' is non-empty: $V' = V''a^qb$ (with $q \geq n+1$). Then $V''a^{q-n-1}\rho V''a^qba^{t-n-1}$. If $t-n-1=0$ we may apply induction. So assume that $1 \leq t-n-1 \leq n$ and then by postmultiplying with b the last ρ-equivalence, we obtain $V''a^{q-n-1}b\rho V''a^{q-n-1}$. Proceeding further in this manner we eventually will arrive at two distinct ρ-equivalent words in their normal forms and then apply induction.

The proof is complete. □

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Department of Mathematics and Statistics, Sultan Qaboos University, Al-Khodh 123, Muscat, Sultanate of OMAN

E-mail address: victor.maltcev@gmail.com